

Completing a right triangle

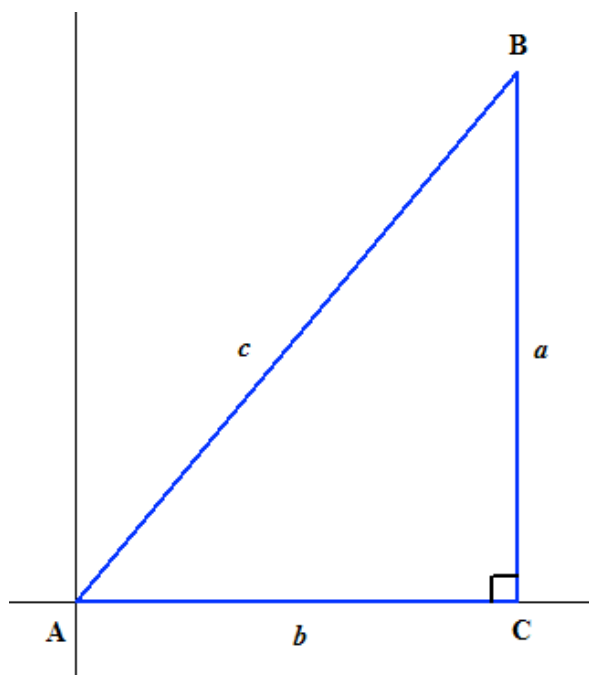
When we talk about “completing a right triangle,” or “solving a right triangle” we mean that we’re going to try to find all three side lengths, and all three angle measures. The good news is that, once you find a few values in the triangle, it gets really easy to find all the rest of the values.

There are a few things we’ll use all the time during this process. First, the three interior angles of any triangle will always sum to 180° , and in a right triangle, one of the angles is 90° , so the other two angles of a right triangle will together sum to 90° as well.

Second, the lengths of the three sides of a right triangle satisfy the Pythagorean theorem.

$$a^2 + b^2 = c^2$$

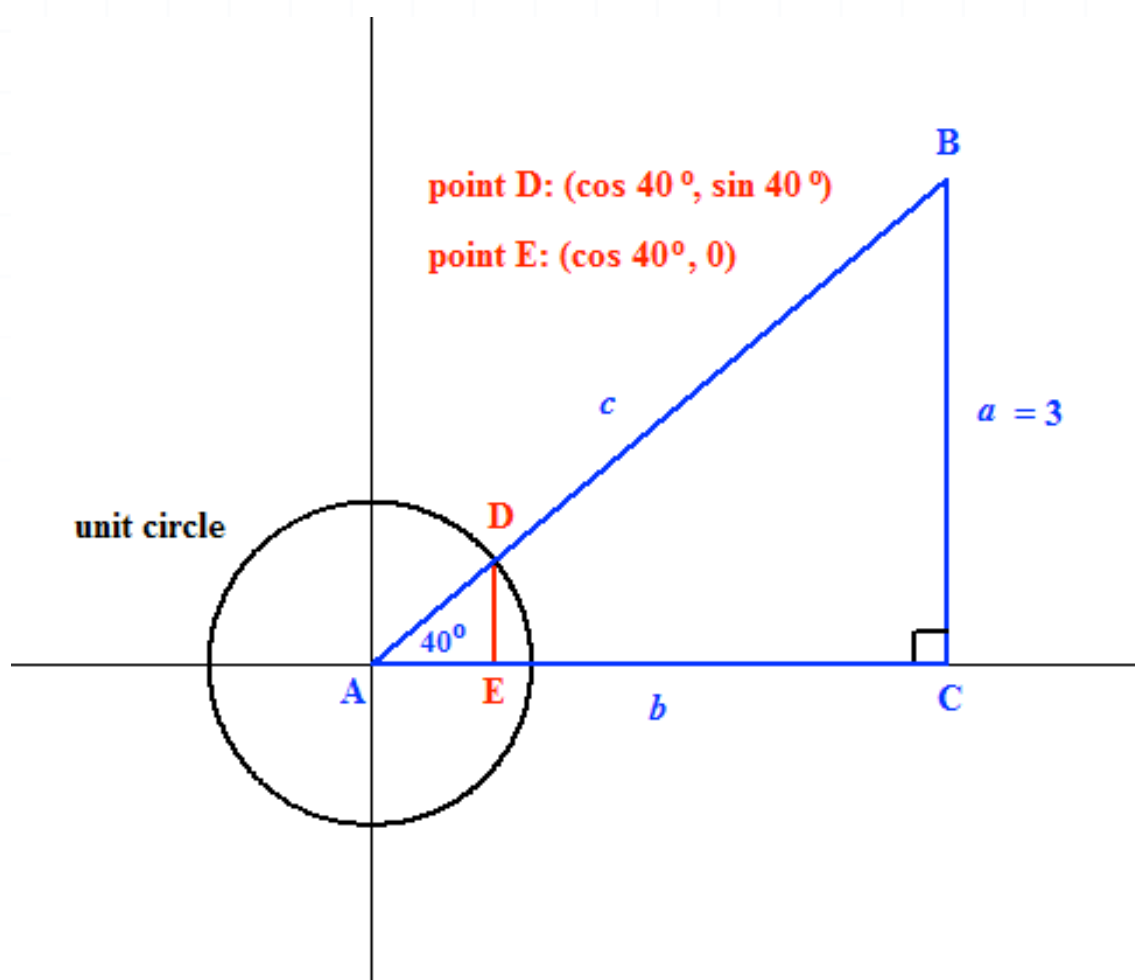
Here, c is the length of the hypotenuse, which is always the longest side and, in a right triangle, the side that’s opposite the right angle. a and b are the lengths of the other two sides and are called the legs.



Example

A right triangle has a leg with length 3. The angle opposite that leg is 40° . Find the measures of all three interior angles and the lengths of all three sides.

It's a good idea to draw a picture of the triangle before we start. Based on what we know, we have



We could just as easily have said $b = 3$ instead of $a = 3$, and therefore, $B = 40^\circ$ instead of $A = 40^\circ$. But it doesn't matter which leg we pick, as long as we make sure that the leg and the 40° angle are opposite each other. It's always helpful to put the right angle out at C so that the hypotenuse can extend out from the origin toward B .



Because it's a right triangle, we have the 90° angle, plus we know we have a 40° angle. Since the interior angles of a triangle always sum to 180° , we already know that the third angle is

$$B = 180^\circ - 90^\circ - 40^\circ$$

$$B = 50^\circ$$

We know the length of side a is $a = 3$, so all we need now is the length of the other two sides. Notice in the diagram we drew earlier that we sketched in $\overline{DE}$ in order to form the smaller triangle ADE . We do that in order to set up similar triangles. We assume that D is on the unit circle, such that $\overline{AD}$ has length 1. Then triangles ADE and ABC are similar, and we can say

Angle DAE is equal to angle BAC . They are both 40° .

Angle ADE is equal to angle ABC . They are both 50° .

Angle AED is equal to angle ACB . They are both 90° .

Therefore, corresponding sides of the two triangles are proportional:

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = \frac{c}{\overline{AD}}$$

With what we already know, we can say

$$\frac{3}{\overline{DE}} = \frac{b}{\overline{AE}} = \frac{c}{1}$$

$$\frac{3}{\overline{DE}} = \frac{b}{\overline{AE}} = c$$



Now here's where the sine and cosine functions come in. Since D is on the unit circle, its coordinates (x, y) are $(\cos 40^\circ, \sin 40^\circ)$. Also, the coordinates (x, y) of E are $(\cos 40^\circ, 0)$. Therefore,

$$\overline{DE} = \sin 40^\circ - 0 = \sin 40^\circ$$

$$\overline{AE} = \cos 40^\circ - 0 = \cos 40^\circ$$

So the proportion becomes

$$\frac{3}{\sin 40^\circ} = \frac{b}{\cos 40^\circ} = c$$

To find b (the length of $\overline{AC}$), we can use

$$\frac{3}{\sin 40^\circ} = \frac{b}{\cos 40^\circ}$$

$$\left(\frac{3}{\sin 40^\circ}\right)(\cos 40^\circ) = b$$

$$b \approx \left(\frac{3}{0.643}\right)(0.766)$$

$$b \approx 3.57$$

To find c (the length of $\overline{AB}$), we can use

$$\frac{3}{\sin 40^\circ} = c$$

$$c \approx \frac{3}{0.643}$$

$$c \approx 4.67$$



So the side lengths are $a = 3$, $b \approx 3.57$ and $c \approx 4.67$, and the angle measures are $A = 40^\circ$, $B = 50^\circ$, and $C = 90^\circ$.

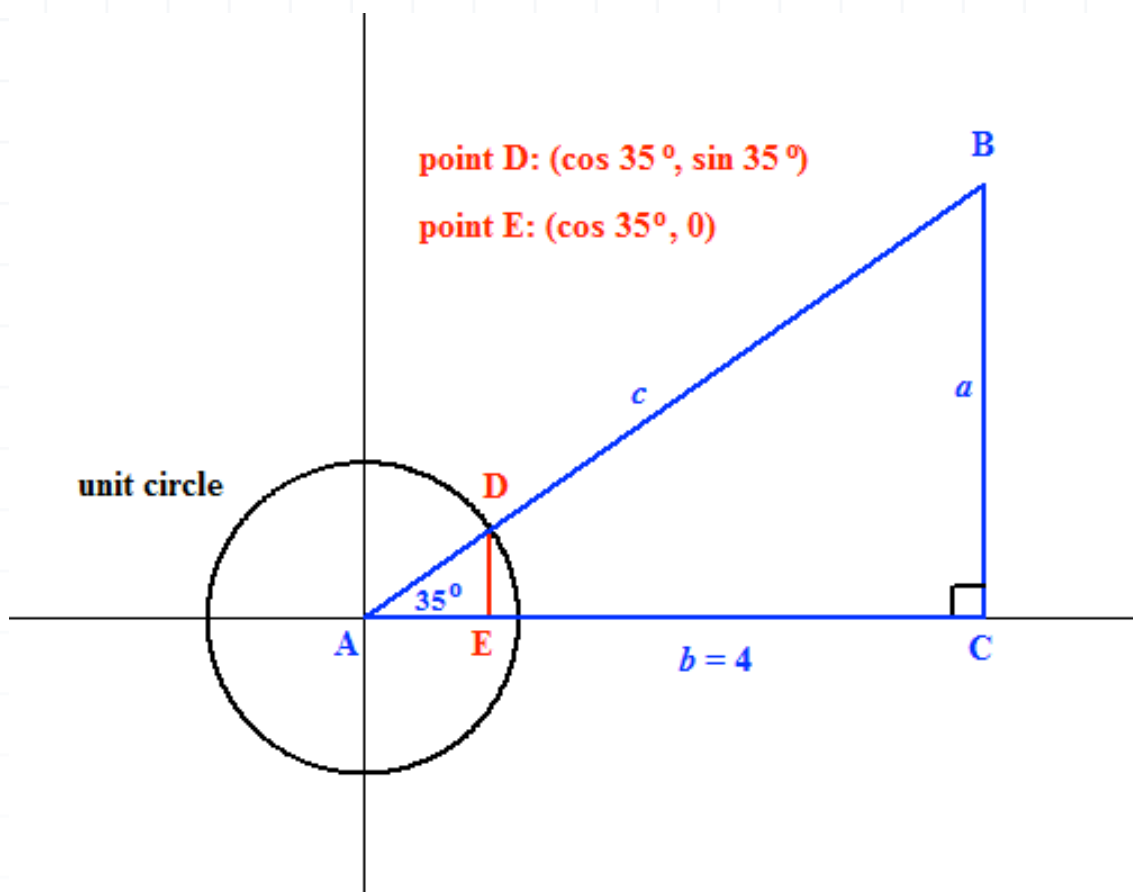
Let's do another example where we have one leg of a right triangle and the angle opposite the other leg.

Example

A right triangle has one leg with length 4 and an interior angle opposite the other leg that's 35° . Complete the triangle.

Let's draw a picture of what we have. Using the same orientation for the triangle as before, we could say $b = 4$ and $A = 35^\circ$, or $a = 4$ and $B = 35^\circ$. Either will work, but we'll use the former. Let's also sketch in the unit circle and put D on the unit circle.





With $C = 90^\circ$ and $A = 35^\circ$, B must be

$$B = 180^\circ - 90^\circ - 35^\circ$$

$$B = 55^\circ$$

Because triangles ABC and ADE are similar, we know

Angles DAE and BAC are both 35° .

Angles ADE and ABC are both 55° .

Angles AED and ACB are both 90° .

Therefore

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = \frac{c}{\overline{AD}}$$



$$\frac{a}{\overline{DE}} = \frac{4}{\overline{AE}} = \frac{c}{1}$$

Point D is again on the unit circle, and its coordinates (x, y) are $(\cos 35^\circ, \sin 35^\circ)$. Also, the coordinates (x, y) of E are $(\cos 35^\circ, 0)$. Therefore,

$$\overline{DE} = \sin 35^\circ - 0 = \sin 35^\circ$$

$$\overline{AE} = \cos 35^\circ - 0 = \cos 35^\circ$$

Plugging this into the proportion, we get

$$\frac{a}{\sin 35^\circ} = \frac{4}{\cos 35^\circ} = c$$

To find a (the length of $\overline{BC}$), we can use

$$\frac{a}{\sin 35^\circ} = \frac{4}{\cos 35^\circ}$$

$$a = \left(\frac{4}{\cos 35^\circ} \right) (\sin 35^\circ)$$

$$a \approx \left(\frac{4}{0.819} \right) (0.574)$$

$$a \approx 2.80$$

To find c (the length of $\overline{AB}$), we can use

$$\frac{4}{\cos 35^\circ} = c$$

$$c \approx \frac{4}{0.819}$$



$$c \approx 4.88$$

So the side lengths are $a \approx 2.80$, $b = 4$ and $c \approx 4.88$, and the angle measures are $A = 35^\circ$, $B = 55^\circ$, and $C = 90^\circ$.

